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Construction of triply periodic minimal surfaces

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We discuss triply periodic minimal surfaces from a mathematical point of view, giving concepts for constructing new examples as well as a discussion of numerical computations based on the new concept of discrete minimal surfaces. As a result we present a wealth of old and new examples and suggest directions for further generalizations.

1. Introduction

Minimal surfaces offer many attractions to diverse disciplines of mathematics and the natural sciences. The reasons for the common interest are the deep problems which appear during closer investigation of their properties, and the widespread application of minimal surfaces in different areas of science. Even in pure mathematics, the research on minimal surfaces has developed into several disciplines concerning different aspects of minimal surface theory and uses very different analytic tools, such as differential geometry and partial differential equations. These differences in research interests may also be seen between, for example, differential geometry and the natural sciences. In differential geometry there have been several great discoveries during the past decade, but since many of them are related to properties such as completeness and finite total curvature, they have not excited much interest in the natural sciences. And on the other hand, problems concerning minimal surfaces, such as the properties of triply periodic minimal surfaces which arose in the natural sciences, have not attracted much attention in mathematics.

The present article is intended to contribute to an interdisciplinary discussion. The purpose is to describe triply periodic minimal surfaces and their properties from a mathematical point of view. This includes new principles for construction and tools for numerical experiments based on discrete techniques. We emphasize that the systematic construction of more and more complicated new examples is possible. This shows that there exists a wealth of triply periodic minimal surfaces, in particular many with the same crystallographic symmetry. We discuss some of the new surfaces we have found and offer directions for further generalization of known examples. Nevertheless, we restrict ourselves to surfaces whose fundamental domain for the translational symmetry group is bounded by planar symmetry lines rather than by straight lines, because the examples bounded by straight lines can be found by crystallographic reasoning rather than by questions of analytic existence.

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Table 1. *List of figures*

In the organization of this paper we use most surfaces as pictorial explanations of the text. To enhance the use of the paper as a (small) library of triply periodic surfaces, a list of figures is included (see table 1). Some tables of surfaces show one-parameter deformations between known surfaces, which reveal a new surface for an intermediate parameter. In this article we can show only very few intermediate steps; for animations we refer to the recent video on minimal surfaces 'Touching soap films', by Arnez *et al.* (1995), which explains minimal surface theory to a popular scientifically interested audience.

In § 2 we review mathematical properties and in § 3 we continue with a discussion of triply periodic minimal surfaces. Section 4 introduces the powerful 'conjugate surface construction', which allows us to construct fundamental domains for most examples of the triply periodic minimal surfaces that are solutions of free boundary problems; this includes even unstable solutions. For practical purposes the numerical construction of minimal surfaces is vital. We include in § 6 an introduction to the concept of discrete minimal surfaces and their properties. For the first time a numerical construction of most known triply periodic minimal surfaces has become an easy task. Specific methods of construction are given in § 5 based on global and local approaches. We close our paper with a picture section.

2. First properties of minimal surfaces

Minimal surfaces have a history of over 200 years. Research began in the middle of the 18th century when, from the research of Lagrange on variational problems, the

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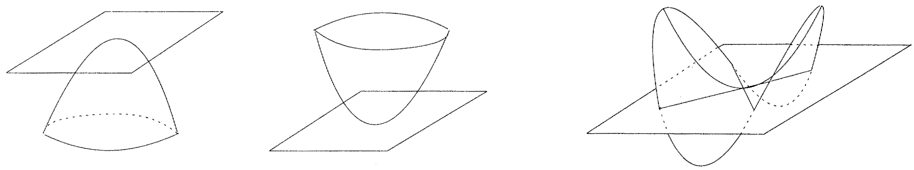

Figure 1. Surfaces are locally like a hill, basin or saddle.

following question arose: 'What does a surface bounded by a given contour look like when it has the smallest surface area?' Lagrange (1867) was interested in general variational problems where minimization and maximization of specific properties must be analysed. The variational problem of minimizing surface area leads to a partial differential equation for the unknown minimal surface, and tools for solving it had not yet been developed. In fact, the first mathematical conjectures about minimal surfaces were derived from the careful observations of soap films by the physicist Plateau (1873). Comprehensive introductions to the historical development of minimal surface theory can be found in the monographs of Nitsche (1989) and Hildebrandt *et al.* (1992).

(*a*) *Local and global definition*

When analysing the properties of surfaces with minimal area we find that smaller pieces of the surface also have minimal area with respect to their own smaller boundaries. A very small neighbourhood of an arbitrary point on such a surface must have minimal area too and must therefore look like a 'saddle'. Around no point can the surface look like a basin or a hill, i.e. it cannot be curved to only one side of the tangent plane, because otherwise the surface area could be reduced by cutting the hill off or filling the basin. A surface which does not stay locally on one side of its tangent plane intersects the tangent plane. A saddle is the best known example for such behaviour, indicated in figure 1.

But minimal surface saddles are even more special: they look the same from both sides. From the soap film point of view this is obvious: since the surface tension of a soap film is in equilibrium at every point, the forces pulling to one side must balance the forces which pull to the other side. In differential geometry the term *mean curvature* measures the bending at a point, which must vanish for minimal surfaces.

For a bigger minimal surface, or physically speaking a bigger soap film, spanned by its boundary, it is not true in general that it has minimal area. There might be other, completely different, soap films with the same boundary but having less area. See figure 2 for example. It is true that each of these soap films locally around each point has minimal area and fulfils the balancing condition. Although a mathematically existing minimal surface is defined by fulfilling the local balancing condition everywhere, the surface might not be realizable as a physical soap film, because the soap film might be globally unstable. We repeat that minimal surfaces are defined to have *locally* minimal area, i.e. *small* portions of a mathematical minimal surface are always realizable as a physical soap film.

(*b*) *Boundary value problems*

The original problem of finding a minimal surface spanned by a given boundary curve is called the *Plateau problem*, named after the Belgian physicist J. A. F. Plateau (1873) who made extensive experimental studies with soap films in the 19th century.

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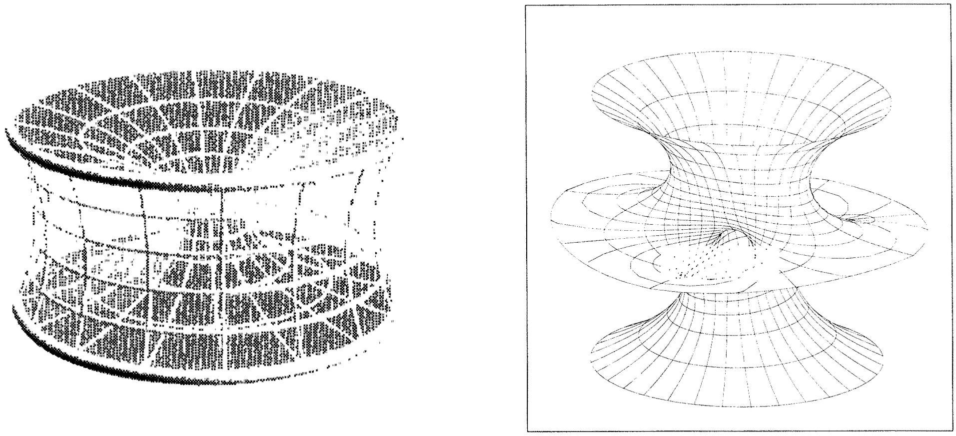

Figure 2. Left: Stable and unstable catenoid (Arnez *et al.* 1995); both surfaces are bounded by the same contour. Right: Costa surface.

A further boundary problem is the so-called *free boundary problem*. Here a part of the boundary curve is restricted to lying on a given plane instead of being a given curve. The boundary is free to choose its position on the bounding plane. A simple argument shows that the minimal surface must meet the boundary plane at a right angle. Otherwise the surface area could be reduced by a simple modification.

Another existence question asks for a surface which is partially bounded by a thread. A thread is characterized as having constant length, fixed end points and no resistance to bending.

In differential geometry one is also interested in infinitely large minimal surfaces without a boundary. Examples are the simple classical catenoid or the recently discovered Costa surface shown in figure 2. The figures show interesting finite portions of the infinite surfaces.

For the construction of minimal surfaces some remarkable properties of boundary value problems can be applied. The most important one is the solubility of the Plateau problem proved by Douglas and Rado:

Theorem 2.1. *Every simple closed boundary curve is spanned by at least one minimal surface.*

This theorem guarantees a minimal patch for every given contour. But the patch is usually not unique; see figure 19.

Today we use this result to also prove the existence of rather specific problems. The emphasis on existence results in mathematics is not always appreciated outside our field. One should keep in mind that so-called physical intuition is no longer a reliable guide if the minimal surface under consideration is unstable. Indeed, for the more general boundary problems mentioned above and for nearly all surfaces considered in this article we do not have such a good existence result as theorem 2.1. This failure is one of the major obstacles to the rigorous construction of triply periodic minimal surfaces. Later we will describe the *conjugate surface method* which helps in many cases to construct solutions for free boundary problems as they occur in the theory of triply periodic minimal surfaces.

(c) *Symmetry and embeddedness properties*

Among the major properties used in construction of new minimal surfaces are the following symmetry properties:

Lemma 2.2. (Straight line) *Every minimal patch bounded by a straight line segment may be extended across the line to a bigger minimal piece: rotate a copy of the original piece by 180° around the straight line. The original piece and the copy will not only have the same tangent plane along the line but also all higher derivatives agree, therefore together both patches form a bigger minimal surface.*

To apply this property choose any polygon in $\mathbb{R}^3$ with angles π/k and which does not cross itself. The Plateau solution of such a polygon can always be extended to an infinite minimal surface in $\mathbb{R}^3$ without boundary. But in most cases this surface will have too many self-intersections to be interesting.

Lemma 2.3. (Planar symmetry) *Every minimal patch which meets a plane orthogonally (e.g. as the result of a free boundary value problem) can be extended by reflection in the plane to a larger patch. As above, in the case of minimal surfaces all derivatives of both patches agree along the plane and therefore form a larger minimal surface.*

The last property is crucial for the construction of the conjugate surface, as we will discuss in greater detail in § 4. But note that one may lose stability when doubling the surface: even if the first piece is a stable minimal surface, the doubled one may not be.

Lemma 2.4. (Point inversion, normal rotation) *If a planar symmetry line and a straight line on a minimal surface meet in a point P, then the inversion $X \to P + (P - X)$ is a symmetry of the minimal surface. If two planar symmetry lines or two straight lines meet under an angle π/k in P, then a rotation by $2\pi/k$ around the normal at P is a symmetry of the surface.*

The last symmetries are important because they persist as the minimal surface is deformed through its associate family. In fact, they may occur without the line symmetries.

So far, outside mathematics, only pictured minimal surfaces have been accepted as existing. In such cases one can see whether they have self-intersection. In mathematics we look for theorems which prove that there are no self-intersections. The problem of self-intersections for periodic minimal surfaces is often decided by a 2-step argument: at first one proves that the fundamental domain is embedded, and then checks that the continuation with the crystallographic symmetry group does not lead to intersecting copies. The arguments for the latter step depend on the symmetry group, and to prove embeddedness of the fundamental domain different techniques may be used. For example the following uniqueness theorem and the theorem of Krust (personal communication; see p. 118 in Dierkes *et al.* 1992) are often applicable:

Lemma 2.5. (i) *If the boundary of a minimal patch has a 1–1 projection onto a convex planar domain then the patch is the unique minimal surface bounded by its contour and, moreover, it is* embedded *(i.e. without self-intersections).*

(ii) (Krust) *The associate family of a minimal patch, which is a graph over a* *convex planar domain, consists of minimal surfaces which are all graphs and therefore embedded.*

3. Triply periodic minimal surfaces

Triply periodic surfaces have by definition translational symmetries in three independent directions. When triply periodic minimal surfaces (TPMS) are considered in the natural sciences, it is almost automatically understood that they are without self-intersections. These are also mathematically the most interesting examples because the ones with self-intersections are so abundant that finding many of them poses no problem.

(a) *Review of known examples*

We first review known surfaces. The most popular examples have the symmetries of a crystallographic group, particularly with a group which is generated by reflections in planes. Such groups have fundamental domains for their translational subgroups which are easy to imagine: convex polyhedra, also called crystallographic cells. The piece of a triply periodic minimal surface inside such a cell is also easy to visualize. It meets the boundary planes of the cell in symmetry lines and it looks like a complicated piece with handles and tunnels, where some of them are opened against the cell's boundary faces.

The most famous of these examples is Schwarz's P-surface (Schwarz 1890) shown in figure 16. Schwarz and his students found five triply periodic surfaces. Incidentally, Meeks (1990) showed that this surface can deformed to have any translational lattice symmetry. Of course, since only very special lattices allow reflectional symmetries, there are usually no symmetries for the deformed minimal surfaces. Meeks's examples without symmetry lines have been ignored, probably because of a lack of available pictures.

Schoen (1970) found many more triply periodic surfaces in crystallographic cells, and made them popular in the natural sciences, but his description as balanced surfaces separating skeletal graphs could not be made into a mathematical existence proof and his work remained unknown among mathematicians.

Later Karcher (1989) proved the existence of Schoen's surfaces using the conjugate surface method. With a refined version of the conjugate surface method Karcher found many more triply periodic examples which are roughly speaking like mixtures of Schoen's examples (Karcher 1990). We will explain the construction method and give several examples in later sections. The method is not restricted to the construction of minimal surfaces in euclidean space $\mathbb{R}^3$; for example, in Karcher *et al.* (1988) and Polthier (1991*a*, *b*) a wealth of periodic minimal surfaces are constructed in $\mathbb{S}^3$ and $\mathbb{H}^3$.

At earlier times pictures of those examples were made from the Weierstraß representation formulas. Such formulas are still unknown for the more complicated examples and deformations, see, for example, the new surfaces in figures 18, 20, 12, 16, 15, 13. Today we have discrete techniques available for experimentation even with these complicated examples. The wealth of existing surfaces should convince us to expect a crystallographic group to have many minimal surfaces.

In the current paper we restrict ourselves to those TPMS whose fundamental domain for the translational symmetry group is bounded by planar symmetry lines, because the mathematical considerations for this class of surfaces are mainly related to analytic questions. Construction of minimal surfaces with polygonal Plateau contours is mainly a crystallographic problem because theorem 2.1 assures the existence of the minimal patch. This approach was taken by Fischer & Koch (1987), who classified the crystallographic groups with enough 180°-rotation axes so that Plateau contours made of pieces of these axes are formed. Fischer & Koch considered boundaries not only for disc-type fundamental pieces but also for annular fundamental pieces.

Two more surfaces deserve especial mention. One is the gyroid of Schoen (1970). It is an embedded triply periodic minimal surface and lies in the associate family of Schwarz's P-surface and D-surface. All the symmetry lines of the P-surface (or the D-surface) correspond to curves on the gyroid that are nearly helices. This explains the name and the difficulty in imagining its shape. Several years ago Lidin & Larsson (1990) found numerically another such surface in the associate family of Schwarz's H-surface. We checked its existence and now refer to it as the Lidinoid. It is as intriguing to look at as the gyroid. Große-Brauckmann & Wohlgemuth (1996) for an embeddedness proof of these surfaces.

(*b*) *Schwarzian chains*

The first examples of TPMS were found by Schwarz (1890). Schwarz was working on the general Plateau problem and he followed the approach of constructing a solution to a modified problem: approximate the boundary contour by a polygon and try to find the minimal surface bounded by the polygon. Then increase the number of sides of the polygon and hope to find a converging sequence. This method did not work in the end, but during this work Schwarz found several other interesting results.

Gergonne (1816) posed the following problem: is it possible to bisect a cube in such a way that the intersection surface is bounded by the inverse diagonals of two opposite faces of the cube, and that the intersection surface has the smallest area? The solution of the problem was to be awarded a prize, received 20 years later by Scherk (1835) for finding similar surfaces – but the Gergonne problem stayed open.

Schwarz used complex analysis and the Weierstraß formula for constructing new surfaces. One of the spectacular results of his methods was the solution in 1865 of the Gergonne problem. He was able to find the Weierstraß functions for many so-called Schwarzian chains. These are mixed boundary contours consisting of straight arcs and planar symmetry lines, i.e. free boundary problems. The contour of the Gergonne problem can be seen as a Schwarzian chain; see figure 10.

4. Conjugate surface method

Over the past decade the conjugate surface method has been established as one of the most powerful techniques for constructing minimal surfaces with a proposed shape. For periodic surfaces the method is very easy to explain and we will do so in this section. We will also mention the difficult aspects when constructing more complicated examples and explain a numerical approach applicable even where theoretical techniques have so far failed.

(*a*) *Associate family of a simple example*

Among the fundamental observations in minimal surface theory in the last century was that every minimal surface belongs to a family of minimal surfaces, the so-called associate family or Bonnet family. Before going into details, the simplest and most popular example is the associate family in which the catenoid deforms into the helicoid. The catenoid is given by

$$C(u,v) = \begin{pmatrix} \cos v \cosh u \\ \sin v \cosh u \\ u \end{pmatrix}$$

and the helicoid by

$$H(u,v) = \begin{pmatrix} \sin v \sinh u \\ -\cos v \sinh u \\ v \end{pmatrix}.$$

With the following weighted sum we obtain the associate family $F^{\varphi}(u,v)$ of both minimal surfaces:

$$F^{\varphi}(u,v) = \cos\varphi \cdot C(u,v) + \sin\varphi \cdot H(u,v).$$

The parameter $\varphi \in [0, 2\pi]$ is the family parameter. For $\varphi = \frac{1}{2}\pi$ the surface is called the *conjugate* of the surface with $\varphi = 0$, and $\varphi = \pi$ leads to a point mirror image. The helicoid is called the *conjugate surface* of the catenoid, and in general each pair of surfaces F^{φ} and $F^{\varphi+\pi/2}$ are conjugate to each other.

Theorem 4.1. *The following properties of conjugate surfaces and the associate family are easily verified by direct computation; see also figure 3 for a pictorial explanation:*

(i) *The surface normals at points corresponding to an arbitrary point* (u_0, v_0) *in the domain are identical, i.e.* $N_{F^{\varphi}}(u_0,v_0) = N_C(u_0,v_0) = N_H(u_0,v_0)$,

(ii) *The partial derivatives fulfil the following correspondence:*

$$F_u^{\varphi}(u_0,v_0) = \cos\varphi \cdot C_u(u_0,v_0) - \sin\varphi \cdot C_v(u_0,v_0),$$
$$F_v^{\varphi}(u_0,v_0) = \sin\varphi \cdot C_u(u_0,v_0) + \cos\varphi \cdot C_v(u_0,v_0),$$

in particular, the partials of catenoid and helicoid satisfy the Cauchy–Riemann equations:

$$C_u(u_0,v_0) = H_v(u_0,v_0), \qquad C_v(u_0,v_0) = -H_u(u_0,v_0).$$

(iii) *If a minimal patch is bounded by a straight line, then its conjugate patch is bounded by a planar symmetry line and vice versa. This can be seen in the catenoid-helicoid examples, where planar meridians of the catenoid correspond to the straight lines of the helicoid.*

(iv) *Since at every point the length and the angle between the partial derivatives are identical for the surface and its conjugate (i.e. both surfaces are isometric) we have as a result, that the angles at corresponding boundary vertices of surface and conjugate surface are identical.*

The last two properties are the most important for the later conjugate surface method.

A more suitable notation for the above relation is given in terms of complex analysis. (Of course, this notation is not essential for our subsequent algorithms and can be overlooked by non-experts.) If we use complex notation then $C + \mathrm{i}H$ is a complex curve in $\mathbb{C}^3$ and with the complex coordinate $z = u + iv$ we have

$$F^{\varphi}(z) = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi} \cdot (C(z) + \mathrm{i} \cdot H(z)) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\varphi} \cdot \begin{pmatrix} \cosh z \\ -\mathrm{i}\sinh z \\ z \end{pmatrix}\right).$$

In a complex analytical description of minimal surfaces the associate family is an easy concept. It is a basic fact in complex analysis that every harmonic map is the real part of a complex holomorphic map. The three coordinate functions of a minimal surface in euclidean space $\mathbb{R}^3$ are harmonic maps. Therefore, there exist three other harmonic maps which define, together with the original coordinate functions, three holomorphic functions, or a single complex vector-valued function. The real part of this function is the original minimal surface, the imaginary part is the conjugate surface, and projections in-between define surfaces of the associate family.

(*b*) *The construction method*

The conjugate surface method has been very successful in solving free boundary problems. As an example, consider the boundary problem for the fundamental patch of the Neovius (1883) surface in figure 3, or with an additional handle as shown in figure 6. If a minimal patch exists with free boundaries at some faces of the tetrahedron as required by the surface, then the conjugate patch in the associate family exists and must be bounded by straight lines. This observation can be reversed to a construction principle: it is sufficient to prove existence of a minimal patch in a corresponding boundary polygon of straight segments (the existence is proved by the solution of the general Plateau problem), then one conjugates it and has a solution of the required free boundary contour. The only remaining problem is to find the corresponding conjugate polygon of straight segments for a given free boundary contour.

To solve this problem two of the properties of minimal surfaces listed above will help us.

1. We know that both patches are isometric, therefore each vertex angle between two adjacent straight segments is identical to the dihedral angle spanned by the two corresponding planes in the free boundary problem.
2. The normal vectors at corresponding vertices are identical. The normal vectors at vertices of the free boundary problem are just the direction of the intersection line of two adjacent boundary planes, and can therefore be read from the free boundary specification.

If the patch has four boundary curves, these two properties uniquely determine the corresponding polygonal contour for a free boundary problem up to scaling. We can now specify detailed instructions for the conjugate surface method:

Theorem 4.2. (Construction of the Polygon). *Given a free boundary problem, i.e. a set of planes which will be met orthogonally by the sought for minimal surface. If this is a well-posed problem we can construct the conjugate contour consisting of straight lines:*

(i) *Read the vertex angles and vertex normals from the arrangement of boundary planes.*

(ii) *Start to generate the conjugate polygonal contour at an arbitrary vertex. Leave the vertex a certain distance along the straight segment, which is orthogonally to the corresponding plane.*

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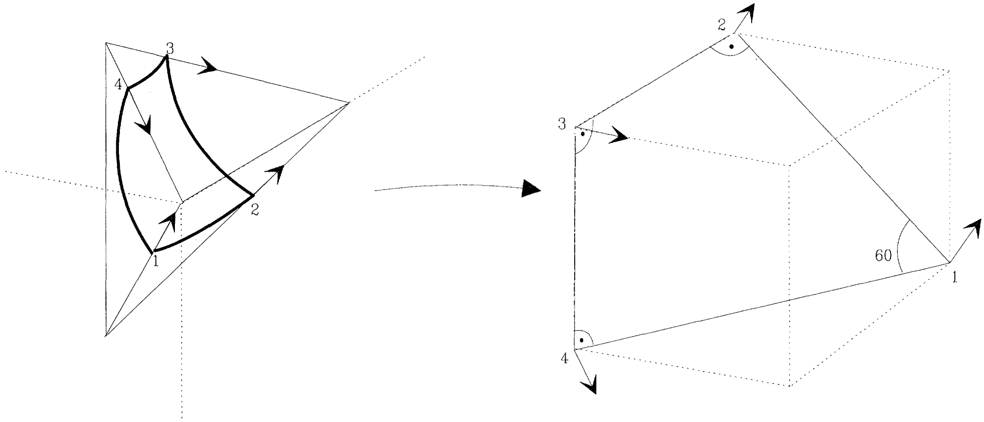

Figure 3. Construction of the conjugate polygonal contour: read vertex angles and vertex normals from the sought for free boundary problem (left) and construct a polygon (right) with the same data. A contour with N edges will lead to $N-4$ unknown edge lengths, i.e. period problems.

(iii) *At the end of the current straight segment repeat the process, but the direction of the next segment is now determined by the vertex normal and the known angle between both straight segments. Repeating this process usually leads to a non-closed polygonal contour.*

(iv) *It remains to adjust the edge lengths such that the contour closes. If the contour consists of four vertices, the edge lengths are uniquely determined up to scaling of the whole contour. For N vertices in the contour one has in general $N-4$ edge lengths to choose, the others are determined by the closing condition.*

(v) *Given the polygonal contour the construction is finished: its conjugated Plateau solution gives a solution for the original free boundary problem.*

A difficulty arises if two boundary curves in the original free boundary problem span the same plane. Since only normal vectors in the above algorithm are used, the construction cannot distinguish between two parallel planes. Therefore, the resulting patch may have all boundary symmetries as required but the two planes are not identical. See figure 6 for a pictorial description of the problem. The problem of having two parallel planes is usually called the *period problem*, because two parallel planes result in an additional unwanted translational period. For more complicated boundary problems the period problem is the major obstacle in existence proofs with the conjugate surface method.

(c) *The period problem*

The so-called period problem is the major obstacle in a successful application of the conjugate surface method. As an introductory example we will try to insert an additional handle in all four horizontal handles of Schwarz's P-surface; see figures 11 and 4.

This was first done by Karcher (1989). We already know the corresponding contour of the fundamental domain of Schwarz's P-surface. Applying the conjugate surface method results finally in the following modification of the polygonal contour: replace the vertex which corresponds to the place where the handle will be inserted with a straight segment orthogonally to the new symmetry plane of the handle as shown in figure 5.

The length of the additional straight segment is equal to the perimeter of the

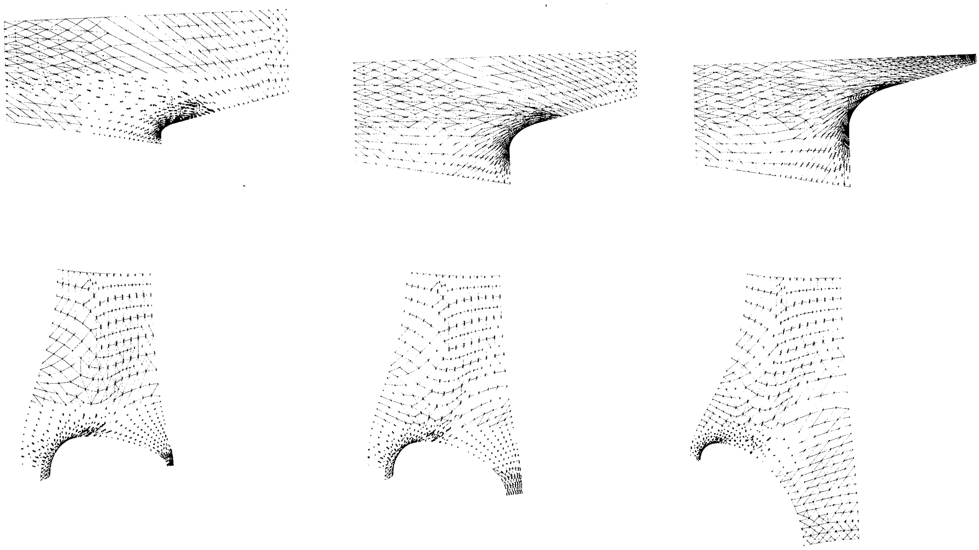

Figure 4. Fundamental domains for the inclusion of handles into Schwarz's P-surface. First line: Plateau surfaces with straight boundary segments. Second line: the desired fundamental pieces conjugate to the above Plateau solutions.

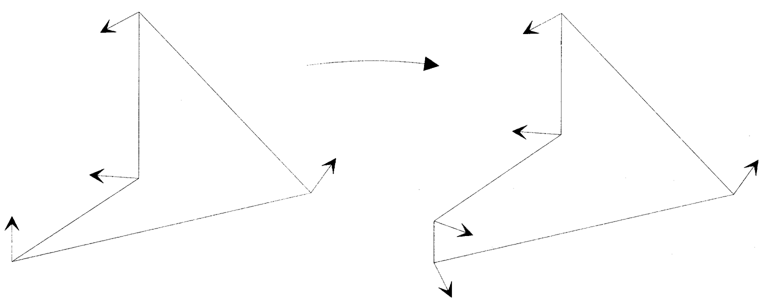

Figure 5. Usually one inserts an additional handle at the vertices of the fundamental domain and chooses the axis of the handle to be parallel to the original surface normal. At the conjugate contour this is done by inserting an additional edge parallel to the surface normal. Compare with next figure.

handle segment. Therefore, a small segment will result in a small handle and the resulting surface is close to Schwarz's original P-surface. After reflection the two handles emanating from top and bottom will not close up. Adjusting the perimeter is necessary so that the rim of the top half-handle coincides with the rim of the bottom half-handle. Otherwise one would be left with two additional symmetry planes and the resulting TPMS would not be embedded.

There is no general theorem to solve the period problem, and, in fact, in many cases there exists no solution. Since a rigorous solution of the period problem is usually a difficult task one must rely for practical purposes and experimental studies on numerical computations. For triply periodic surfaces one often does not know the Weierstraß representation functions, but one can use the recently developed concept of discrete minimal surfaces.

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5. The concept of handle insertion

A very successful method in the construction of minimal surfaces is the *concept of handle insertion.* Many new examples and modifications of known surfaces were constructed by using this method. Our pictures only show a small sample from this rich collection. In the following we will explain the concept in detail and give a step-by-step application of the conjugate surface method in the case of a one-parameter problem.

Let us start with some heuristics. Consider a soap film with a large flat region, for example, occurring on minimal surfaces where two planar symmetry lines meet at an angle of π/k, $k > 3$. A relatively large neighbourhood around such a point is a stable minimal patch inside its boundary and one can expect to some extent that a tiny modification of the surface at such a point should not disturb the whole minimal surface too much. For example, let us put a small ring onto the soap film at such a point and cautiously pull it off the surface. As long as the film does not burst we see a small handle developing. See, for example, figure 14, where at the centre of every face of the I-Wp surface small handles develop to become the O,C-TO surface. Surely, the heuristic example is not quite correct, since local modification of existing minimal surfaces immediately modifies the surface even at points far away.

But, as the example of the O,C-TO surface shows, one can expect to control the construction in some cases. In the following we will assume that the boundary of a new handle is a planar symmetry line, since we want to construct triply period minimal surfaces. Also, it will turn out that for inserting new handles it is not necessary to restrict them to flat areas.

Let us take the fundamental patch of an existing minimal surface, e.g. the surface of Neovius. This patch is bounded by the faces of a tetrahedron. We specify a modified free boundary problem by searching for a minimal patch with an additional boundary curve on the tetrahedron, which will later result in a new handle at the centre of each cubical face of Neovius's surface. The original free boundary solution for the fundamental patch of Neovius's surface was already an unstable patch, and this is even more true for the more complicated patch we have in mind now. Therefore neither computing the patch numerically with a direct minimization algorithm will work without further additional tricks, nor will easy theoretical concepts directly prove the existence of such a patch.

At this point the conjugate surface method can show its full advantage. Instead of constructing the patch directly we construct its conjugate patch. The conjugate patch is bounded by straight lines as we have seen in § 4 and we can compute its polygonal contour from the information we get from the way in which the sought for patch must lie inside the tetrahedron; see figure 6:

(i) The patch and polygonal contour must have the same vertex angles; therefore we read from the tetrahedron that the polygonal contour must have vertex angles 45°, 90°, 90°, 90° and 60° at corresponding vertices.

(ii) In addition we know that the normal vectors at corresponding vertices are identical and that every polygonal segment is orthogonal to the plane of its corresponding planar arc.

(iii) Let us draw the first vertex of the polygonal contour and its normal. Then we reach out in one of the two directions orthogonally to the plane of the first planar arc; in figure 5 we have chosen to go down. We must assume an arbitrary length for the first arc, and stop somewhere to define the second vertex.

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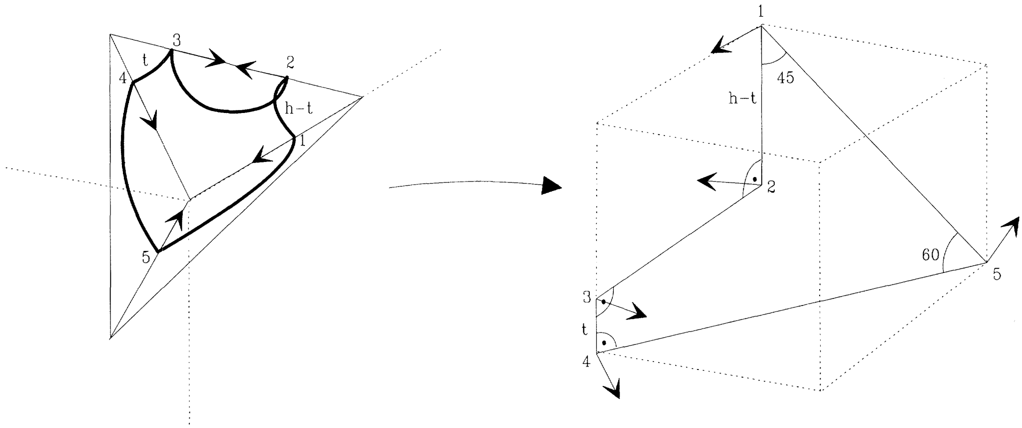

Figure 6. A standard period problem: the new handle inserted at vertex 2 in figure 3 adds a new symmetry arc between vertex 1 and 2 which must be equal to the existing symmetry arc between vertex 3 and 4. The parameter t varies between 0 and h, thereby making the handle smaller resp. bigger.

(iv) Again, let us mark at that point the second vertex and draw its normal vector. Now the direction in which we must leave to the third vertex is uniquely given by the vertex angle and the normal vector.

(v) At every step we assume an arbitrary edge length and continue with this procedure until we reach the last vertex, number five in our example. The condition that the polygonal contour must close uniquely determines the pairwise ratios of the second, fourth and fifth edge lengths. The first and third edge lengths may instead take on two arbitrary values which must sum up to the total height h of the cube. h may be normalized to 1, other values will only scale the minimal surface.

Let us define the length of edge three as our parameter t, then length of edge one is determined as $h - t$. That means, we have a 1-parameter family of possible contours. Each contour will lead to a minimal patch with required symmetries, but only one will lie in the tetrahedron (in fact, uniqueness is not guaranteed). We shall now go on to construct the minimal surface patch.

(i) Every polygonal contour of the 1-parameter family bounds a minimal surface, its Plateau solution.

(ii) Conjugating the 1-parameter family leads to a family of surfaces which all have the required symmetries.

(iii) If for some parameter values the period is negative and for others positive, then there exists an intermediate parameter value t_0 with vanishing period. This can be seen if one can show that the two limit surfaces corresponding to parameter values $t = 0$ and $t = 1$ exist and have opposite sign in their limit period: for $t = 0$ we have Neovius's surface, i.e. the inserted handle is too small (period is negative), and for $t = 1$ we have Schwarz's surface whose handle is far too big (period is positive).

(iv) The surface corresponding to t_0 is the final fundamental patch for Neovius's surface with additional Schwarz handles whose existence we have just proved.

The period can be explicitly measured, it is in our case the distance between the symmetry plane of the top of the inserted handle and the parallel symmetry plane of the Neovius handle. If both symmetry planes are not identical then reflection in both planes will generate an additional translation orthogonal to the planes. Therefore the resulting surface will have self-intersections as long as the period does not vanish.

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In this existence proof we used the continuous dependence of Plateau solutions on their boundary contour, i.e. the period depends continuously on the polygonal contour. In full generality this is false, but for our surfaces the family of contours can be parallel projected to the boundary of a fixed convex domain. In such cases a result of Nitsche assures continuity (Nitsche 1965).

(a) *Other handle types*

There are different ways to look at handles. Their name suggests that handles are small but we have seen already that they may become big and dominate the surface. Therefore, there is often some ambiguity when referring to handles. But from the constructive point of view it is clear what a handle means.

Classically, a handle is a cylindrical connection between two objects. If one considers two adjacent cells of Schwarz's P-surface, then both are connected by such a classical handle. The way we construct handles, the handle is symmetric w.r.t. its waist. We call this handle a *Schwarz handle.*

Increasing the symmetry of the classical handle leads to the Neovius handle type, which consists of half handles centred at a point and forming a regular star. The handles of a Neovius surface form such a configuration around the edges of the bounding cube.

Similarly one can increase the symmetry to three dimensions and obtain handles which point in the directions of all faces of a Platonic solid. They are called I-Wp handles since the fundamental cell of the I-Wp surface of Schoen forms such a handle with octahedral symmetry (the I-Wp surface is delicate, since every building block of the handle has a rotational symmetry exchanging its ends, therefore one can become confused, because the fundamental cell of the I-Wp surface is bounded by a cube. But one should observe how the handle parts of the I-Wp surface are connected in the centre of the cube). It is best to name these kinds of handles by the Platonic solid they correspond to.

6. Discrete minimal surfaces and numerics

Numerical computations of minimal surfaces may be done with several different techniques. Classically, most examples were computed by using the following two methods. The first assumes knowledge of the Weierstraß integration formulas and does a numerical integration. For complicated surfaces this is a non-trivial task, but the method has never failed in such cases up to now. The second method relies on finite-element theory and tries to solve the underlying partial differential equations with numerical techniques. Both methods have drawbacks when applied to the computation of triply periodic minimal surfaces: the first method needs the Weierstraß formula functions which are difficult to derive for more complicated examples, and the second method relies on energy minimization techniques, which will also have difficulties, since fundamental domains of more complicated examples are usually not stable minima for a known energy functional. They are only critical points, and therefore direct minimization will usually miss these examples. To deal with this problem one needs to construct additional restrictions for each case.

In this section we will introduce a different method based on discrete techniques. *Discrete* means that surfaces are not considered as smooth objects but, for example, as a combinatorial complex of triangles. A similar approach is also made in finite-element theory, but there one always has a smooth limit surface in mind, which would

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be obtained when the discretization level approaches zero. The instability of the solution of the free boundary problem is avoided because the discrete approach can handle the conjugate surface method. The derivation of Weierstraß data is avoided (and often impossible), since the discrete minimal surface patches will be defined by their polygonal contours.

Let us start with a definition of a discrete minimal surface. For simplicity, we restrict ourselves here to triangulated discrete surfaces, but the reader should keep in mind that already during the conjugate surface construction discrete surfaces with other combinatorial structure occur:

Definition 1. A *discrete surface* is a collection of triangles which have the structure of a topological simplicial complex, i.e. any two triangles are either disjoint or have a single edge in common or a single point.

This definition covers ordinary triangulated surfaces. But it also includes, for example, edges where several triangles meet as it occurs in experiments with soap foam. This is out of the scope of the present paper.

Let us now refine the above definition to the case of area minimizing discrete surfaces. The area of discrete surfaces is defined to be the sum of the areas of each individual triangle. But as in the case of smooth minimal surfaces we make the definition more general and include those surfaces which minimize area only locally (i.e. which may globally not be area minimized):

Definition 2. A *discrete minimal surface* is a discrete surface with the property that no single vertex of the triangulation can be moved to decrease its area, i.e. the surface is a critical point for the discrete area functional.

It turns out that the minimality condition at every vertex can be described by an explicit formula in terms of geometric scalars. Consider figure 7 where the local neighbourhood of a point on a discrete surface is shown.

Lemma 6.1. (Balancing condition) *The following formula describes a balancing condition that every discrete minimal surface fulfils. The condition should be seen in analogy to the surface tension of soap films which balances at every point. Mathematically the relation describes the vanishing of the gradient of the area at a point p. It must be fulfilled at every point p if the surface is a discrete minimal surface:*

$$\frac{\partial}{\partial p}\,\text{area(triangulation)} = \sum_{i=1}^{\#\text{neighbours of } p} (\cot\alpha_i + \cot\beta_i)(p - q_i) = 0. \tag{6.1}$$

The formula can also be interpreted as a weighted sum of the edges $p-q_i$ emanating at p. The weighting factors $\cot\alpha_i + \cot\beta_i$ are computed by using the cotangent of the two angles which lie in the two adjacent triangles opposite to the edge. Figure 7 explains this pictorially. This interpretation explains the analogy to the tension of a soap film that also balances at every point. The analogy to the smooth situation can be driven further by defining the vector-valued sum as the equivalent of the smooth Laplacian, i.e. as the discrete Laplacian. Like the smooth case here the discrete Laplacian must vanish for discrete minimal surfaces.

A remarkable and unexpected fact is the simplicity of the minimality condition. Of course, the same formula must come out when using finite-element theory, but there

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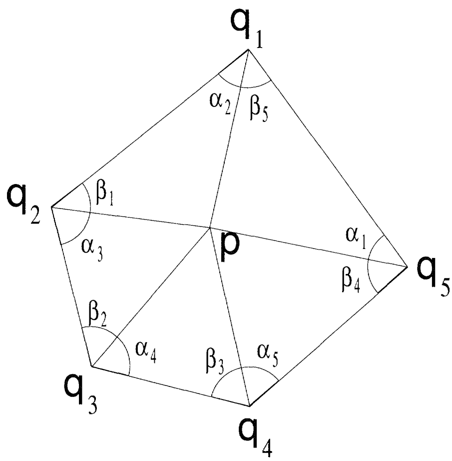

Figure 7. Neighbourhood around a point on a discrete surface.

one would not see the influence of the geometric terms like angles and edges in such a clear way. It would be hidden behind integrals of finite-element basis functions. Also we have here the opportunity to go further and to conjugate the discrete minimal surface. This is not a trivial task and was done by Pinkall & Polthier (1993). The fact that *conjugation* can be defined for discrete minimal surfaces in such a way that similar properties to the smooth case remain true should be considered as a substantial success of the discrete concept and the definition of discrete surfaces. We do not go into the details of the conjugation algorithm in this paper but refer to the reference above.

(*a*) *Solving stable and unstable problems*

In numerical practice we do not solve the above equilibrium condition (6.1) directly but use instead an iteration process based on a different energy functional, as described in detail in Pinkall & Polthier (1993). At the boundary the gradients are restricted in such a way that the boundary conditions are always fulfilled during minimization. This allows us to apply the minimization algorithm also to free boundary problems. In such a case the gradient is projected onto the plane of the boundary, thereby restricting motion of the boundary points in the required way. But solving free boundary problems with a direct minimization approach only succeeds if the resulting minimal patch will be stable. For most of the more complicated surfaces this is not the case: here their fundamental patch is an instable minimal patch, i.e. when we apply a numerical method to find the minimal patch experimentally, then it will be minimized further and will usually degenerate to an edge of the bounding polyhedron; see figure 8. In such instable cases the numerical conjugate surface method is still applicable if the conjugate contours have stable Plateau solutions.

(*b*) *Numerics of the period problem*

When working with the conjugate surface method in most cases the conjugate contour is only known up to a number of free parameters. These parameters are the lengths of some edges of the polygonal boundary. Wrong values of the edge lengths will result in so-called periods in the final minimal surface, as explained in detail in § 5. For simplicity consider the case of only one free parameter, i.e. one period which

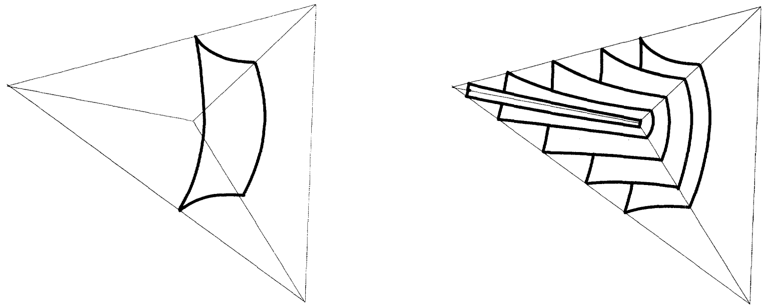

Figure 8. The fundamental pieces of Schwarz's P, Neovius's and I-Wp surfaces considered as free boundary value problems in a tetrahedron are unstable surfaces. Minimizing their energy leads to degeneration as shown in the right-hand figure.

must be controlled. If one chooses the corresponding edge length in the polygonal contour to be too small, the period will be, let's say, negative, and if it is too large the period will be positive. A zero-length period is known to exist inbetween by continuity arguments. Numerically we choose a set of different edge lengths which are in some way distributed among the possible edge lengths and compute the surface and its conjugate for all these lengths. Then we apply an interpolation technique between the resulting surfaces and obtain for some parameter value a vanishing period.

At first sight this sounds quite trivial, but in practice we need several concepts to cope with the problems that occur. For example, one usually applies adaptive refinement of the triangulations at regions with high curvature as it can be seen in the family of the O,C-TO surface in figure 14. Such situations already require techniques to interpolate among a sequence of surfaces with different triangulations.

(*c*) *Numerical sample session*

Finally, let us briefly list the necessary steps to compute a minimal surface with our program:

(i) Specify the major vertices of the boundary contour (i.e. four vertices for a quadrilateral) and the type of boundary curves in a definition file.

(ii) Load the definition file into the program and invoke the automatic surface builder which generates a triangulated surface inside the boundary contour. The number of triangles can be chosen interactively by adaptive refinement depending on the surface curvature.

(iii) Invoke the minimization algorithm to compute the corresponding discrete minimal surface.

If the contour specification belonged to the conjugate minimal patch one must now continue with:

(iv) Apply the conjugation algorithm.

In the original boundary specification one may already mark boundary vertices so that they may depend on further parameters like, for example, edge lengths. This allows specification of contours which depend on one (or more than one) parameter, i.e. a family of boundary curves. The program automatically generates initial contours for several parameter values and then applies all operations to the family as a whole, i.e. the user works as above; the only difference is that minimization takes more time since finitely many surfaces of the family must be minimized. At the end the user must interactively check where the period is closed.

(v) Check vanishing periods in surface family.

Additional operations might be invoked in the case of more complicated examples. This includes, for example, adaptive refinement, reflection of the resulting minimal patch to a larger surface, or computation of the boundary of the crystallographic cell.

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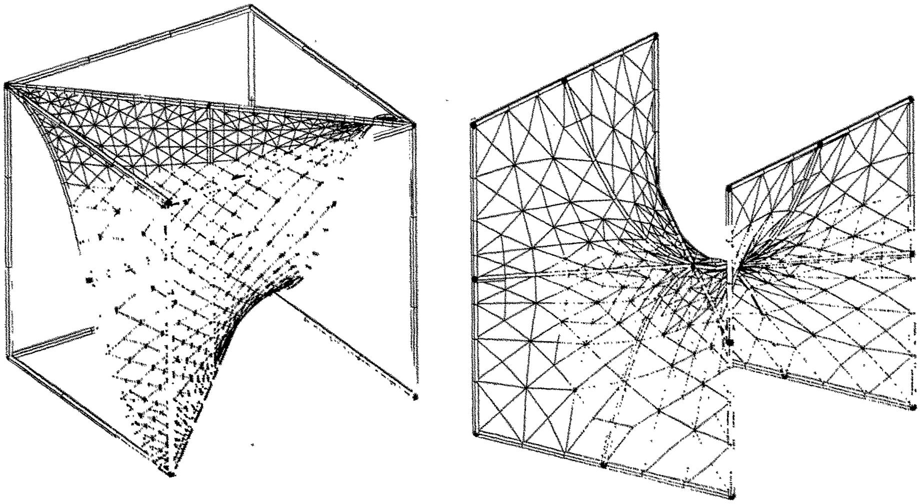

Figure 9. Original Gergonne problem (left) and assembled like Scherk's doubly periodic solution with ends cut off (this shows that Scherk was close to a solution.)

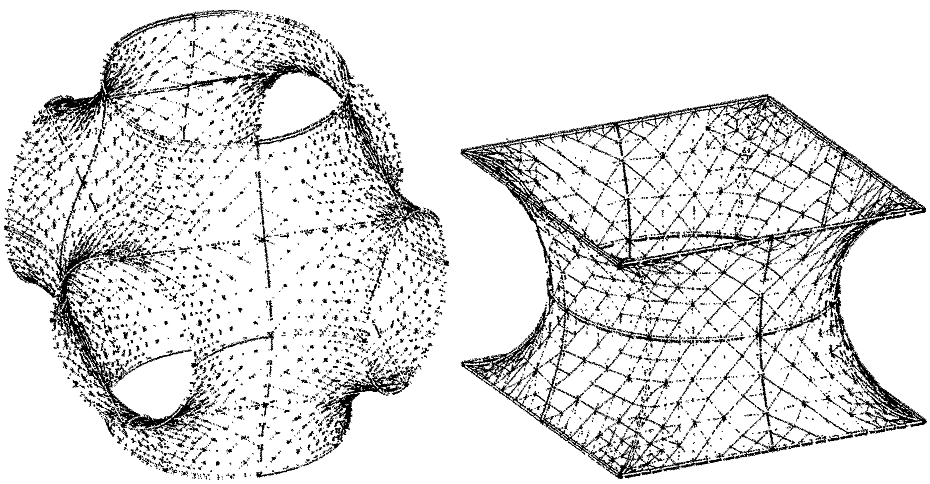

Figure 10. Different aspects of the conjugate surface of the Gergonne solution (Schwarz's P-surface is obtained if diagonal of the original Gergonne contour in figure 9 (left) is chosen of length twice that of the vertical edge).

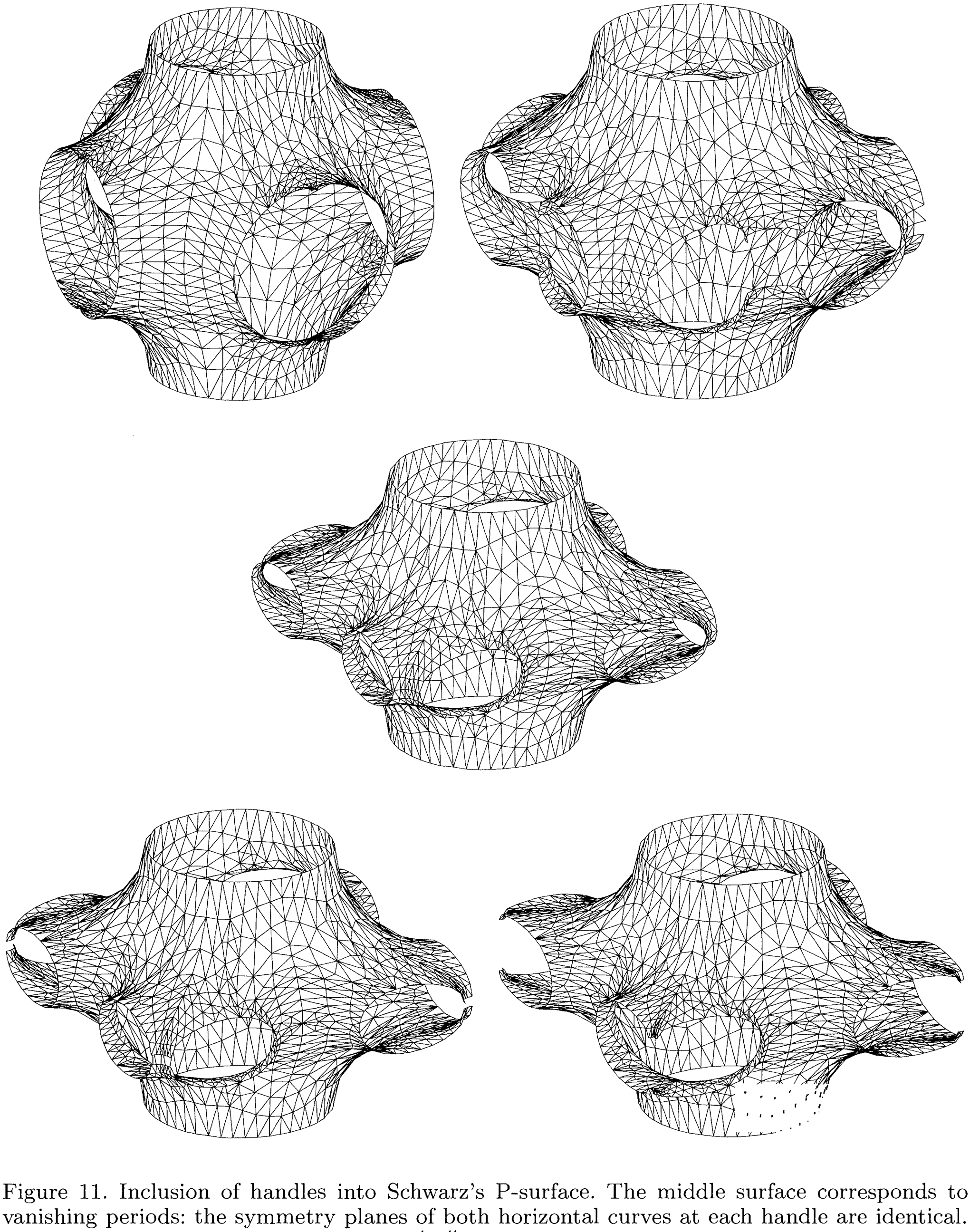

Figure 11. Inclusion of handles into Schwarz's P-surface. The middle surface corresponds to vanishing periods: the symmetry planes of both horizontal curves at each handle are identical. The deformation continues to Schoen's S′-S″ surface, bottom right.

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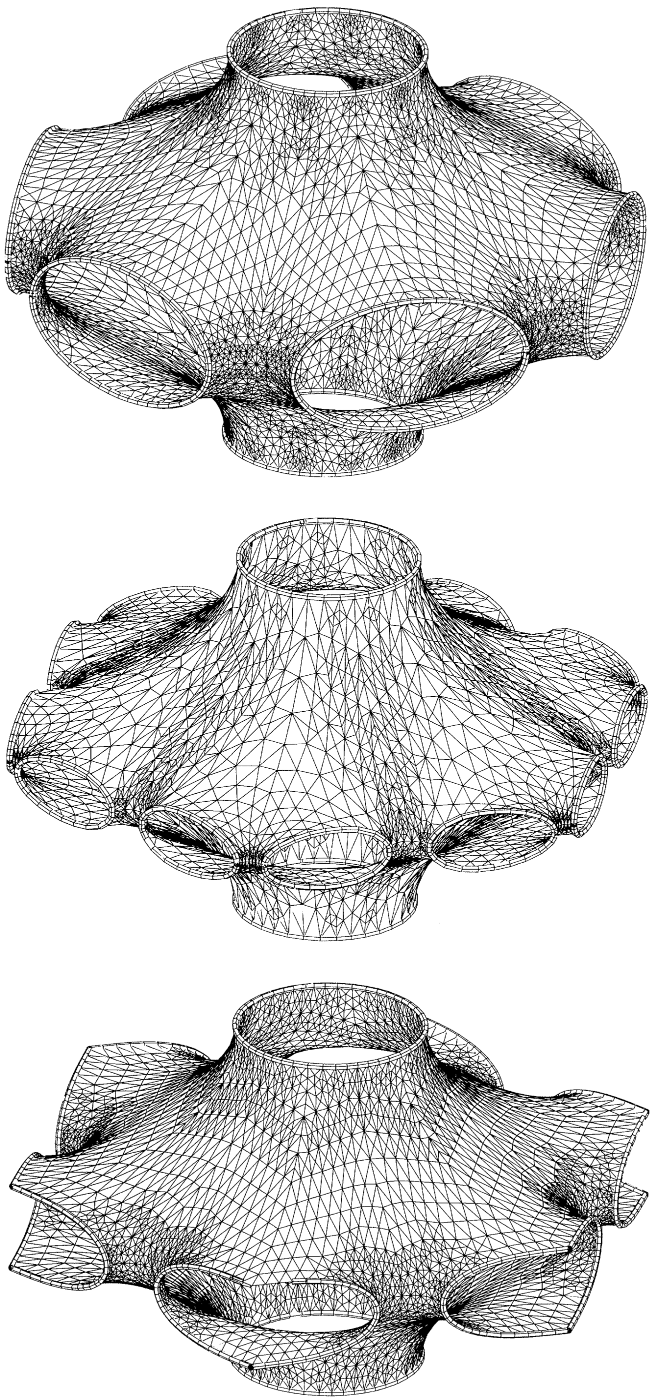

Figure 12. Deformation of H″-T surface into H″-R surface with an intermediate surface.

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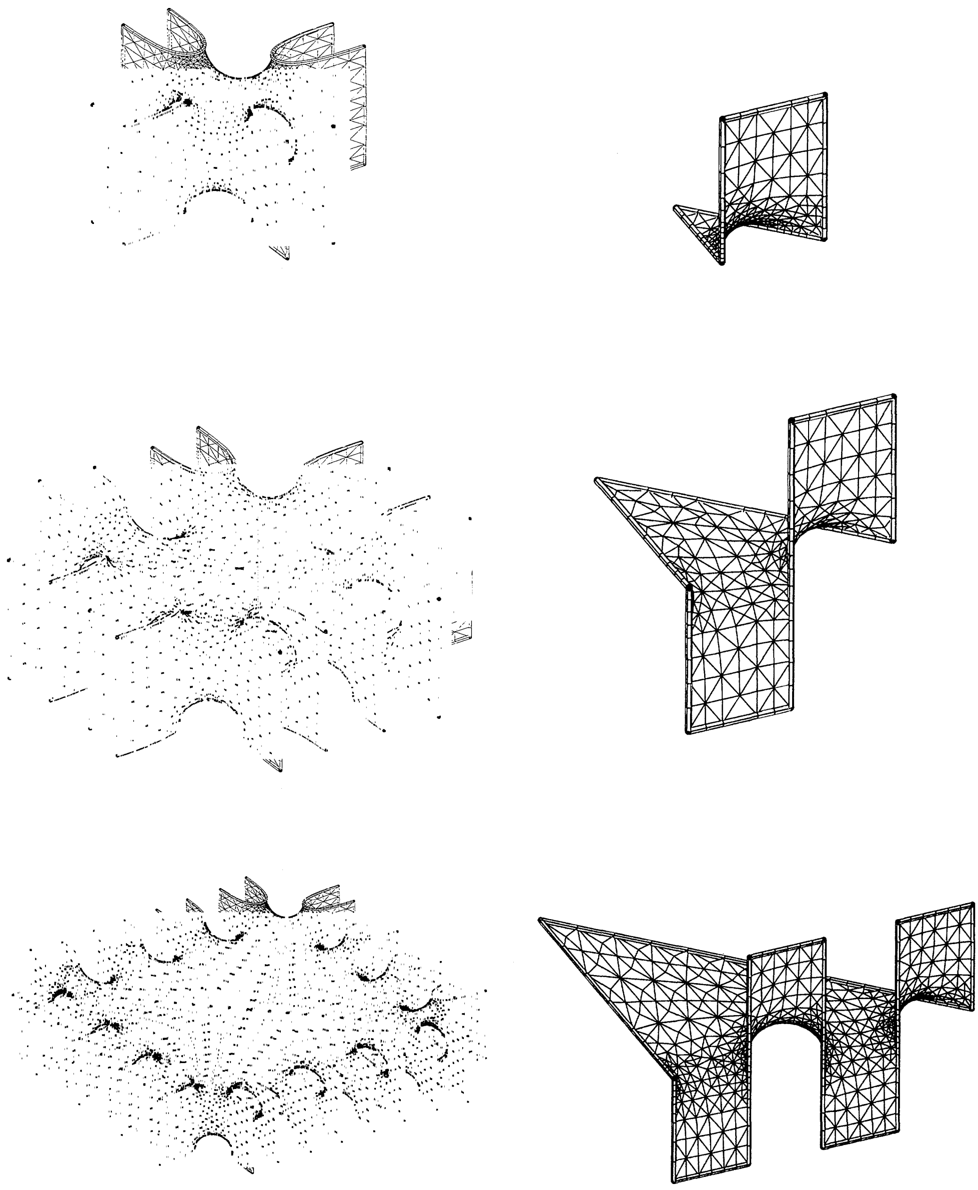

Figure 13. The surface family exists numerically for fundamental polygons with arbitrarily many wings. With each additional wing the genus of the fundamental block in the cube increases. Therefore, this surface family is a candidate for periodic surfaces where the fundamental block has abitrary high genus. A detailed study of this surface is shown in a sequence of the video 'Touching soap films'.

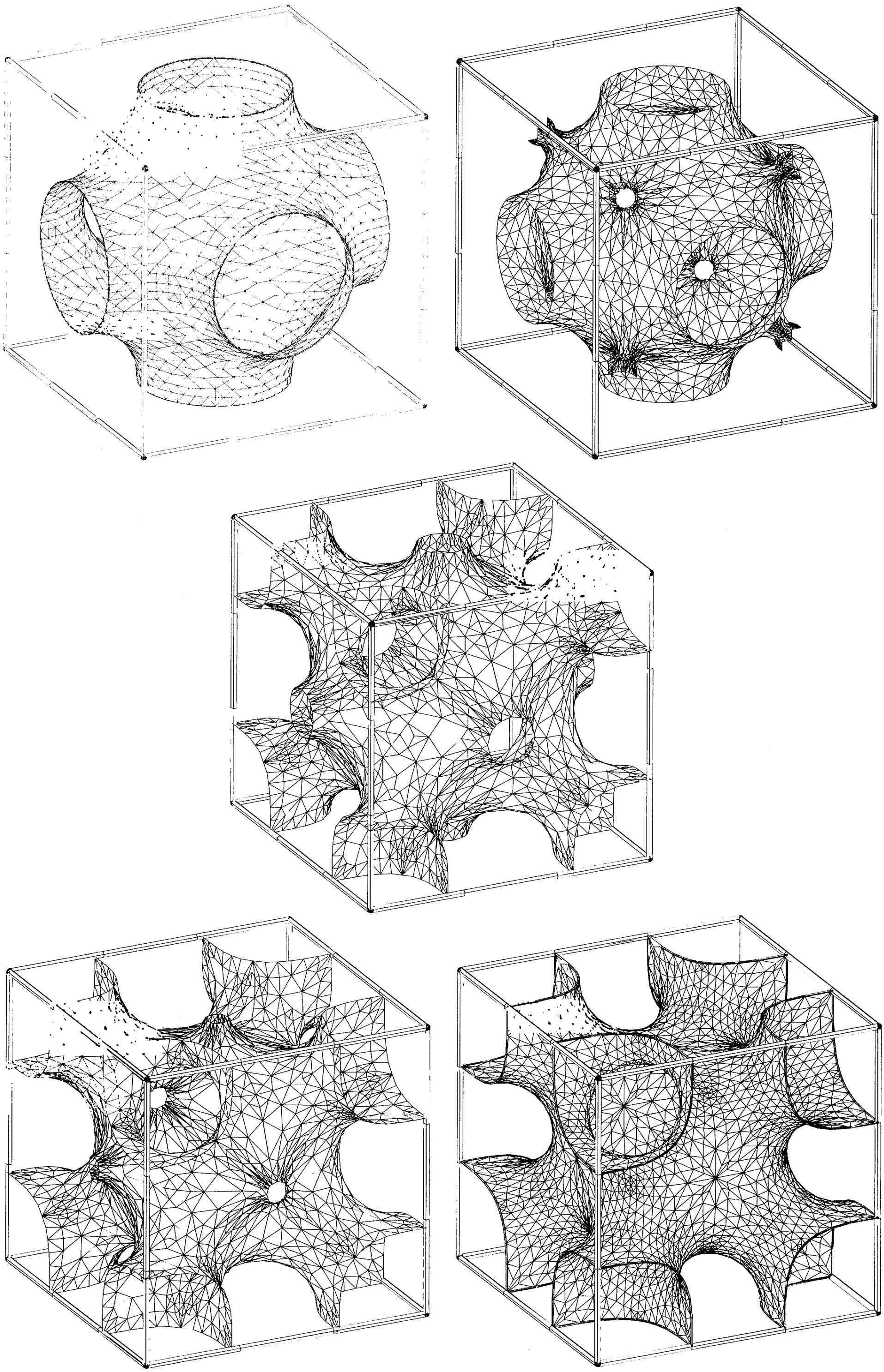

Figure 14. Deformation of Schwarz's P-surface into the I-Wp surface with an intermediate surface O,C-TO. For the O,C-TO surface no simpler existence proof is known than the one indicated by this family.

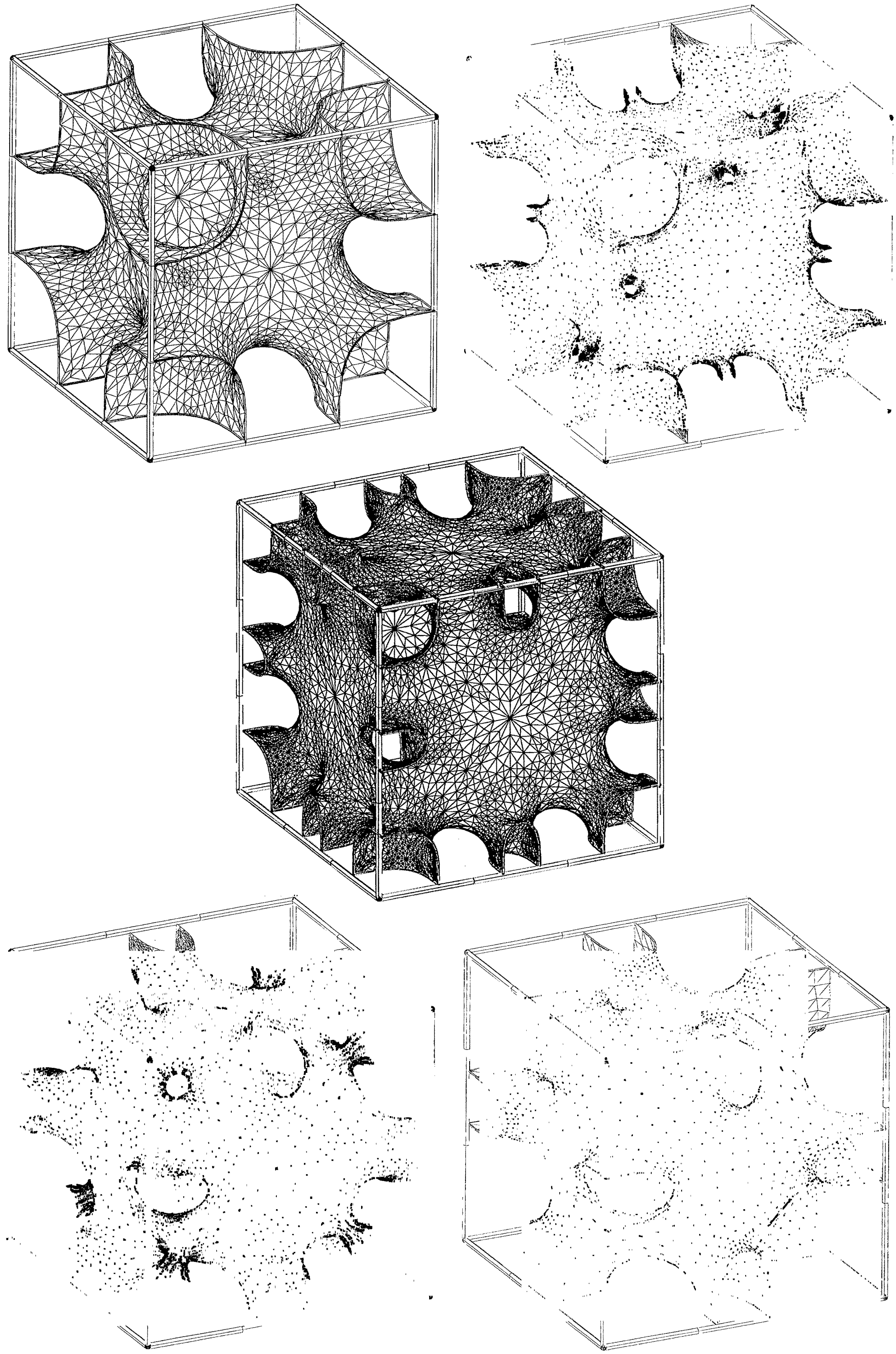

Figure 15. Deformation of the I-Wp surface to Neovius's surface with an intermediate surface. Obviously, the intermediate divides the cube into two regions with unequal volumes.

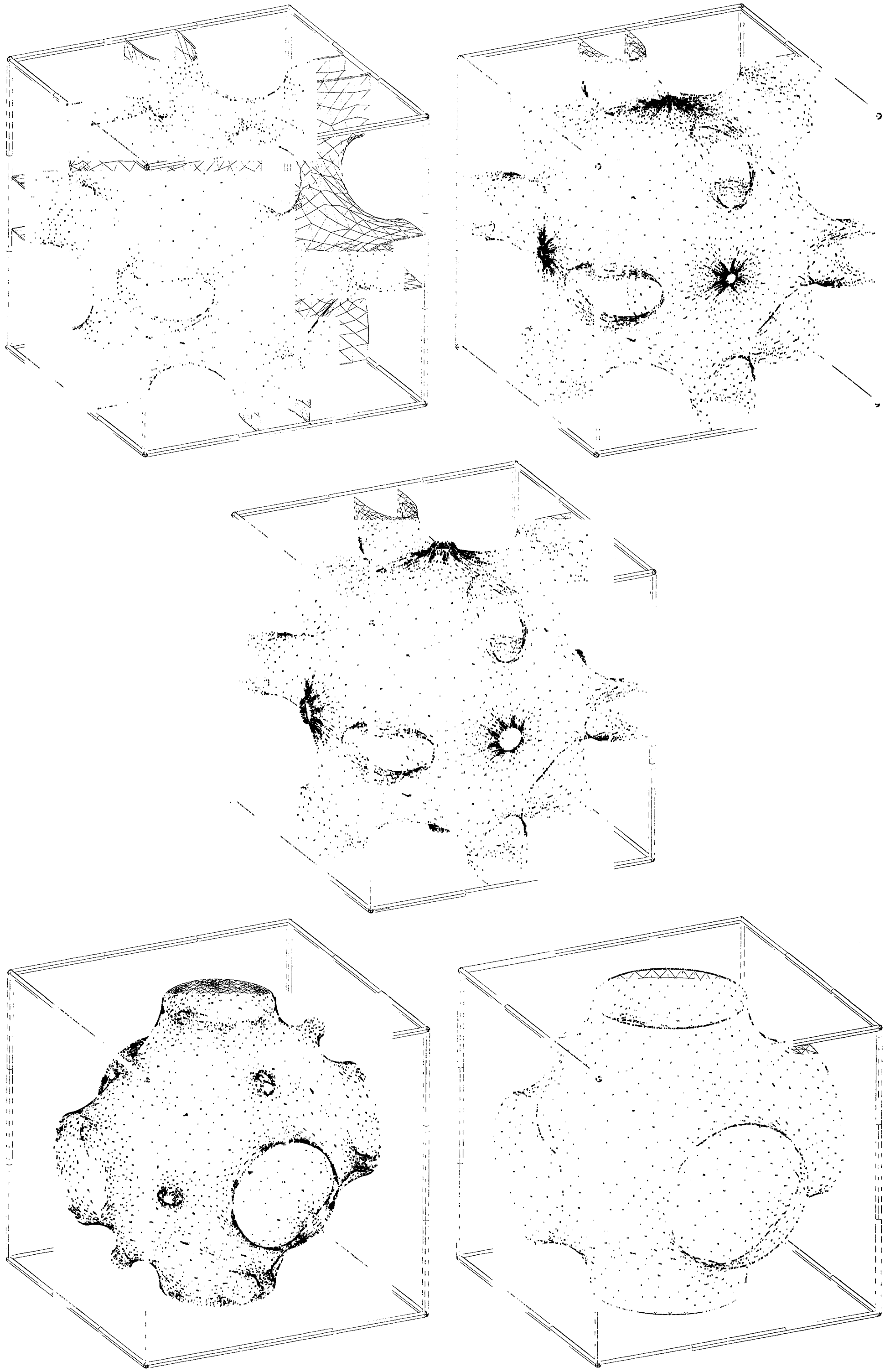

Figure 16. Deformation of Neovius's surface to Schwarz's P-surface with an intermediate surface.

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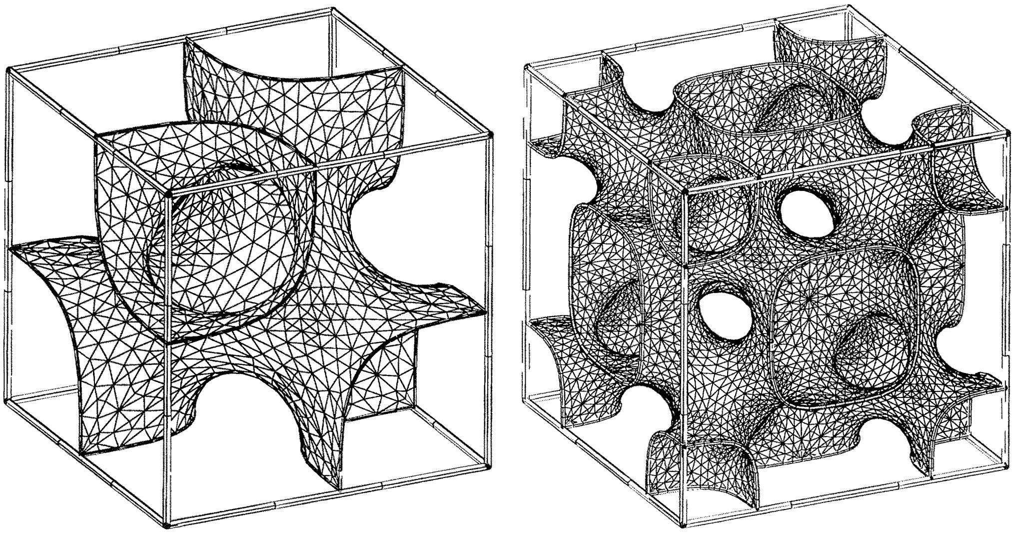

Figure 17. The F-Rd surface was found by A. Schoen. The left figure shows half of a translational fundamental piece fitted into a cube. Four of these blocks can be viewed as the O,C-TO surface with a handle built into the interior from every edge of the bounding cube (right).

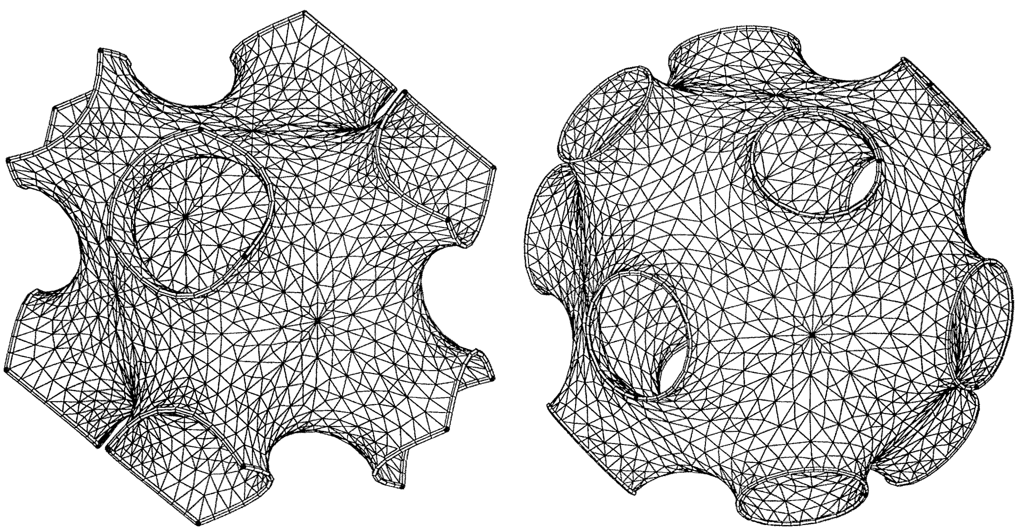

Figure 18. Two fundamental domains of the F-Rd surface for the translational symmetry group. All figures on this page are different views of the same surface.

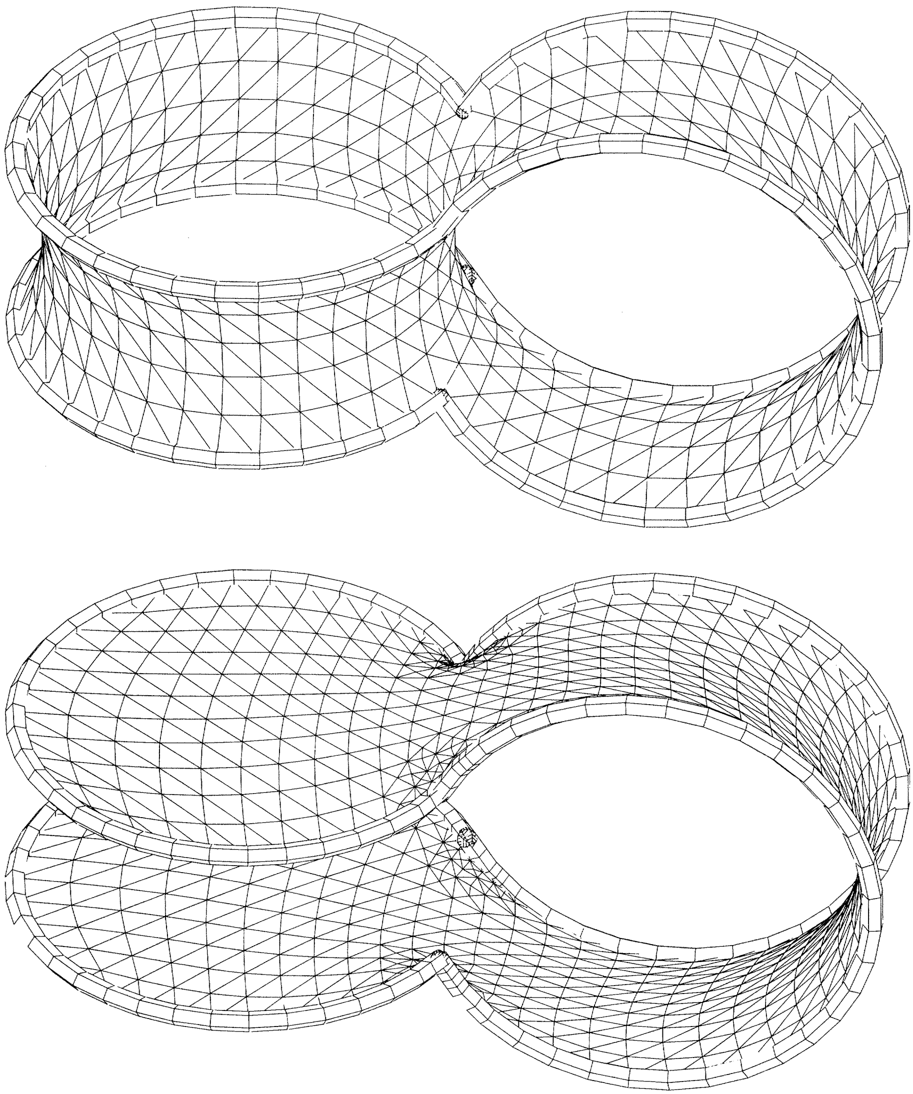

Figure 19. One contour bounding two different stable minimal surfaces.

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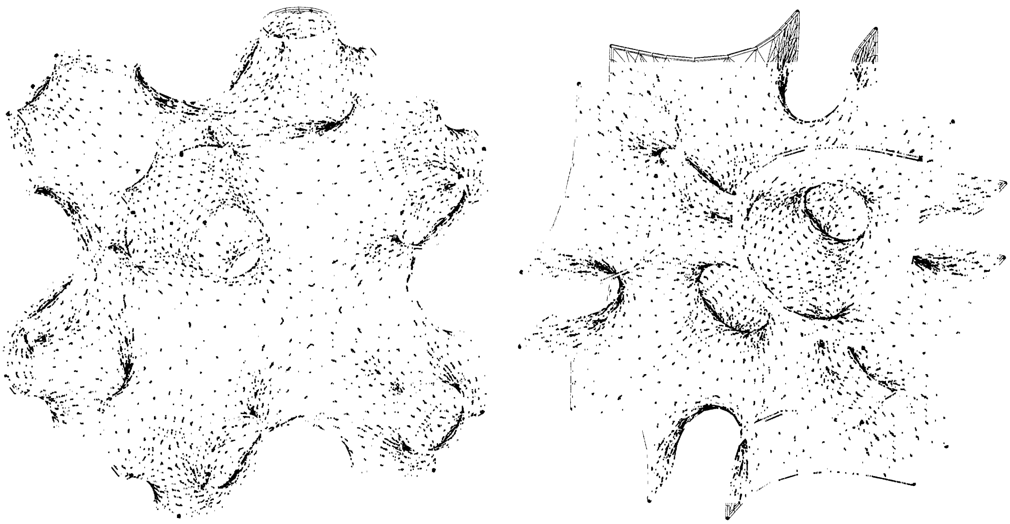
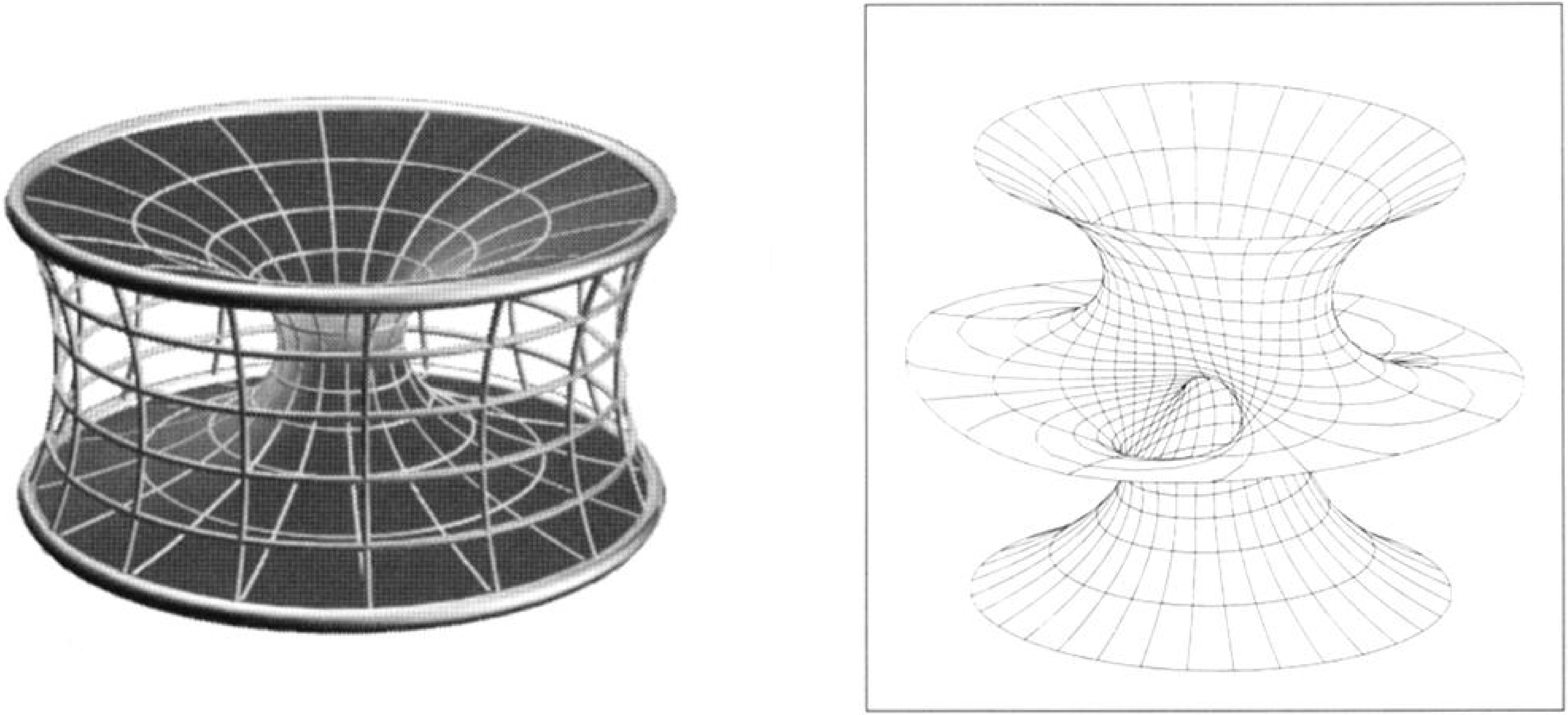


Figure 20. Left: surface with fundamental block in a cube with four handles at every face of the cube. The surface can be interpreted as a modification of Schwarz's P-surface. Right: same surface assembled differently. It can now be interpreted as the I-Wp surface with additional holes to the interior at all faces of the bounding cube.



gure 2. Left: Stable and unstable catenoid (Arnez *et al.* 1995); both surfaces are bounded by the same contour. Right: Costa surface.